MATH 347H: FUNDAMENTAL MATHEMATICS, FALL 2017

PRACTICE PROBLEMS FOR MIDTERM 1

1. Prove that for all sets A, B,

(a) $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$.

(b) $(A \cup B) - B = A - B$.

2. Read the following definition and analyze it.

Definition. Vectors $\vec{v}_1, \vec{v}_2 \dots, \vec{v}_k \in \mathbb{R}^n$ are called *linearly independent* if

$$\forall a_1, a_2, \dots, a_k \in \mathbb{R}\Big[a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k = \vec{0} \implies \big(\forall i \le k, a_i = 0\big)\Big].$$

Vectors $\vec{v}_1, \vec{v}_2 \dots, \vec{v}_k \in \mathbb{R}^n$ are said to be *linearly dependent* if they are not linearly independent.

(a) Write out explicitly what it means for vectors $\vec{v}_1, \vec{v}_2 \dots, \vec{v}_k \in \mathbb{R}^n$ to be linearly dependent. The only negation sign/word in your sentence should be negating equality $\neq$.

(b) Are the vectors $(1,1)$ and $(1,0)$ linearly independent? Prove your answer.

(c) Are the vectors $(1,0,0)$, $(0,1,1)$ and $(1,1,1)$ linearly independent? Prove your answer.

3. (a) For sets X, Y, define what is a function $f : X \to Y$.

(b) Let $f \subseteq \mathbb{R} \times [0,+\infty)$ be defined as the set of all pairs $(x,y) \in \mathbb{R} \times [0,+\infty)$ such that
- if $x \le -1$ then $y = x^2$,
- if $-1 \le x \le 1$ then $y = |x|$,
- if $x \ge 0$ then $y = x$.

Is f a function $\mathbb{R} \to [0,+\infty)$? Justify your answer.

4. Let X be a set and recall that $\mathscr{P}(X)$ denotes its powerset. Recall the operation of symmetric difference $A \triangle B$ and realize that it is a *binary operation* on $\mathscr{P}(X)$, i.e. it is a function $\mathscr{P}(X) \times \mathscr{P}(X) \to \mathscr{P}(X)$ that takes a pair (A,B) of subsets of X to $A \triangle B$.

(a) Verify that $\triangle$ (taken in place of $+$) satisfies Axioms (A2) and (A3).

(b) Show that (A4) also holds by finding a set $\mathbb{O} \in \mathscr{P}(X)$ that serves as the *identity* for the operation $\triangle$, i.e. is such that, for any set $A \in \mathscr{P}(X)$, $A \triangle \mathbb{O} = A = \mathbb{O} \triangle A$.

(c) Show that even (A5) holds by finding, for each $A \in \mathscr{P}(X)$, a set $A' \in \mathscr{P}(X)$ such that $A \triangle A' = \mathbb{O} = A' \triangle A$.

5. Do 1.5.7 and 1.5.9 of Sally's book.

6. Prove that $2^n \geq 2n$ for every $n \in \mathbb{N}$.

7. (a) For a set X, what is a binary relation on X? Write down the definition.

(b) Let R be any rectangle in $\mathbb{R}^2$. Is R a binary relation on $\mathbb{R}$?

(c) For an arbitrary binary relation S on $\mathbb{R}$, determine what geometric (in terms of its shape) conditions on S correspond to reflexivity, irreflexivity, symmetry, and anti-symmetry.